M340 Solutions for in class Exam 2

For each of the following statements, decide if it is true or false and give a brief explanation for your answer.

1. The matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ has an eigenvalue whose algebraic multiplicity is 3 but the geometric multiplicity is 1

FALSE. The algebraic multiplicity is clearly 3 but,

$$A - 4I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad rank = 1,\ \dim N(A - 4I) = 3 - 1 = 2$$

$x_3 = 0$ but x_1, x_2 are free. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= x_1\vec{E}_1 + x_2\vec{E}_2$$

i.e., there are 2 independant e-vects, $\vec{E}_1$ and $\vec{E}_2$ so geom mult=2

2. The vector valued functions $\vec{X}_1(t) = [e^{4t}, 0, 0]^\top$ and $\vec{X}_2(t) = [0, e^{4t}, 0]^\top$ are solutions for $\vec{X}'(t) = A\vec{X}(t)$ if A is the matrix from problem 2.

TRUE (plug into $\vec{X}'(t) = A\vec{X}(t)$ and see)

3. The 3 by 3 matrix A has distinct real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$. Then the general solution for $\vec{X}'(t) = A\vec{X}(t)$ is $C_1e^{\lambda_1 t}\vec{E}_1 + C_2e^{\lambda_2 t}\vec{E}_2 + C_3e^{\lambda_3 t}\vec{E}_3$ and for any initial condition $\vec{X}(0) = \vec{X}_0$ there is a unique set of constants C_1, C_2, C_3 for which the initial condition is satisfied. The fundamental matrix for the matrix for A is

$$M(t) = \left[e^{\lambda_1 t}\vec{E}_1,\ e^{\lambda_2 t}\vec{E}_2,\ e^{\lambda_3 t}\vec{E}_3\right]$$

and $M(t)$ is invertible for all t.

TRUE

Since the e-vals are distinct, the e-vects are linearly independant.
In this case $C_1e^{\lambda_1 t}\vec{E}_1 + C_2e^{\lambda_2 t}\vec{E}_2 + C_3e^{\lambda_3 t}\vec{E}_3$ is the general solution for the ODE
Since the e-vects are LI, they form a basis for R^3 which means for any $\vec{X}_0$ in R^3
there is a unique set of constants C_1, C_2, C_3 for which $C_1\vec{E}_1 + C_2\vec{E}_2 + C_3\vec{E}_3 = \vec{X}_0$
Since $e^{\lambda_1 t}\vec{E}_1,\ e^{\lambda_2 t}\vec{E}_2,\ e^{\lambda_3 t}\vec{E}_3$ are all solutions of the same system and they are LI at $t = 0$, they are LI for all t. Then $M(t)$ has LI colums for all t so it is invertible for all t

4. The general solution for $\vec{X}'(t) = A\vec{X}(t) + \vec{F}(t)$ is the sum of a solution for the homogeneous equation plus a particular solution of the form $\vec{X}_p(t) = M(t)\vec{C}(t)$ where $M(t)$ is the fundamental matrix for A and $\vec{C}(t)$ is obtained by solving $M(t)\vec{C}'(t) = \vec{F}(t)$.

TRUE

If $\vec{X}_H(t)$ is a homogeneous solution and $\vec{X}_p(t)$ is a particular solution, then

$$\frac{d}{dt}\left(\vec{X}_H(t) + \vec{X}_p(t)\right) - A\left(\vec{X}_H(t) + \vec{X}_p(t)\right) = 0 + \vec{F}(t)$$

Using variation of parameters to find $\vec{X}_p(t)$ means assuming $\vec{X}_p(t) = M(t)\vec{C}(t)$
Substituting $\vec{X}_p(t) = M(t)\vec{C}(t)$ into the inhomogeneous system leads to the equation $M(t)\vec{C}'(t) = \vec{F}(t)$ for the unknown $\vec{C}(t)$.

5. The only critical point for the dynamical system

$$u'(t) = -u(t) - v(t)$$
$$v'(t) = u(t) - v(t)$$

is a spiral source at $(0,0)$.

FALSE

The system is linear and so has just one CP at $(0,0)$.
To classify this point we find

$$J = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

which has e-vals $\lambda = -1 \pm i$. Then the CP is a spiral sink.

6. If A is an n by n matrix and $A\vec{E} = \lambda\vec{E}$, then $e^{At}\vec{E} = e^{\lambda t}e^{(A-\lambda I)t}\vec{E} = e^{\lambda t}\vec{E}$.

TRUE

$$\begin{aligned} e^{At}\vec{E} &= e^{\lambda t}e^{(A-\lambda I)t}\vec{E} \\ &= e^{\lambda t}[\vec{E} + t(A-\lambda I)\vec{E} + \cdots] \\ &= e^{\lambda t}\vec{E} \text{ since } (A-\lambda I)\vec{E} = \vec{0} \end{aligned}$$

7 If A is a 2 by 2 matrix with complex eigenvalue $\lambda_1 = \alpha + i\beta,$ and corresponding eigenvector $\vec{E}_1 = \vec{U} + i\vec{V},$ then

$$\vec{Z}(t) = e^{\alpha t}\{\cos\beta t\vec{U} - \sin\beta t\vec{V}\}$$

is one real valued solution for $\vec{X}'(t) = A\vec{X}(t)$.

TRUE

$$
\begin{aligned}
\vec{X}(t) &= e^{\lambda_1 t}\vec{E}_1 = e^{(\alpha+i\beta)t}\left[\vec{U}+i\vec{V}\right] \\
&= e^{\alpha t}\{\cos\beta t + i\sin\beta t\}\left[\vec{U}+i\vec{V}\right] \\
\mathrm{Re}\left\{\vec{X}(t)\right\} &= e^{\alpha t}\left\{\cos\beta t\vec{U} - \sin\beta t\vec{V}\right\} \\
&= \vec{Z}_1(t)
\end{aligned}
$$

Then $\vec{Z}_1(t)$ is real valued and solves $\vec{Z}_1'(t) = A\vec{Z}_1(t)$.

8. If A is a 2 by 2 matrix with complex eigenvalues $\lambda_1,\ \lambda_2,$ and corresponding eigenvectors $\vec{E}_1, \vec{E}_2$ then

$$\vec{X}(t) = e^{\lambda_1 t}\vec{E}_1$$

is a complex valued solution for $\vec{X}'(t) = A\vec{X}(t),$ and the real part and imaginary part of $\vec{X}(t)$ are two linearly independent real valued solutions for $\vec{X}'(t) = A\vec{X}(t).$

TRUE

$$
\begin{aligned}
\vec{X}(t) &= e^{\lambda_1 t}\vec{E}_1 = e^{(\alpha+i\beta)t}\left[\vec{U}+i\vec{V}\right] \\
&= e^{\alpha t}\{\cos\beta t + i\sin\beta t\}\left[\vec{U}+i\vec{V}\right] \\
&= e^{\alpha t}\left\{\cos\beta t\vec{U} - \sin\beta t\vec{V}\right\} + ie^{\alpha t}\left\{\cos\beta t\vec{V} + \sin\beta t\vec{U}\right\} \\
&= \vec{Z}_1(t) + i\vec{Z}_2(t)
\end{aligned}
$$

Since $\vec{X}'(t) = A\vec{X}(t),$ we find

$$
\begin{aligned}
[\vec{Z}_1(t) + i\vec{Z}_2(t)]' &= A\left[\vec{Z}_1(t) + i\vec{Z}_2(t)\right] \\
\vec{Z}_1'(t) - A\vec{Z}_1(t) &= -i[\vec{Z}_2'(t) - A\vec{Z}_2(t)] = 0
\end{aligned}
$$

9. The eigenvectors of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$ form a basis for R^3.

TRUE. The matrix is symmetric so the eigenvalues are real and the eigenvectors are mutually orthogonal. Therefore they form a basis for R^3

10. The matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ satisfies $(A-I)^2 = [0].$ Therefore

$$e^{At} = e^t e^{(A-I)t} = e^t[I + t(A-I)] = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

FALSE. The matrix satisfies $(A-2I)^2 = [0]$ so

$$e^{At} = e^{2t}e^{(A-2I)t} = e^{2t}[I + t(A-2I) + \cdots] = e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$